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Surface-wave group-delay and attenuation kernels

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SUMMARY

We derive both 3-D and 2-D Fréchet sensitivity kernels for surface-wave group-delay and anelastic attenuation measurements. A finite-frequency group-delay exhibits 2-D off-ray sensitivity either to the local phase-velocity perturbation $\delta c/c$ or to its dispersion $\omega(\partial/\partial\omega)(\delta c/c)$ as well as to the local group-velocity perturbation $\delta C/C$. This dual dependence makes the ray-theoretical inversion of measured group delays for 2-D maps of $\delta C/C$ a dubious procedure, unless the lateral variations in group velocity are extremely smooth.

Key words: attenuation, Fréchet derivatives, global seismology, Q, sensitivity, surface waves.

1 INTRODUCTION

Increasing theoretical attention has been paid in recent years to the limitations of JWKB ray theory as a basis for inverting surface-wave dispersion measurements. Finite-frequency sensitivity kernels that account for the ability of a mantle Love or Rayleigh wave to 'feel' 3-D structure off an unperturbed great-circle ray have been developed by a number of investigators (e.g. Snieder 1986; Snieder & Nolet 1987; Yomogida & Aki 1987; Friederich *et al.* 1993; Friederich 1999; Spetzler *et al.* 2002; Ritzwoller *et al.* 2002; Yoshizawa & Kennett 2002). The most complete finite-frequency surface-wave analysis to date is that of Zhou *et al.* (2004), hereinafter referred to as ZDN04. In that paper we used the Born approximation to derive Fréchet kernels expressing the sensitivity of surface-wave phase-delay, arrival-angle and amplitude measurements to 3-D variations in the fractional P-wave velocity $\delta\alpha/\alpha$, S-wave velocity $\delta\beta/\beta$ and density $\delta\rho/\rho$. In this paper we tie up two loose ends left hanging by ZDN04; specifically, we derive both 3-D and 2-D sensitivity kernels for group-delay as opposed to phase-delay measurements (e.g. Levshin *et al.* 1992, 2001; Ritzwoller & Levshin 1998; Ritzwoller *et al.* 2002; Shapiro & Ritzwoller 2002; Maceira *et al.* 2005) and for measurements of anelastic attenuation (e.g. Romanowicz 1990; Durek *et al.* 1993; Romanowicz 1994, 1995, 1998; Billien *et al.* 2000; Selby & Woodhouse 2000, 2002; Gung & Romanowicz 2004). This paper is not self-contained; in the interest of brevity we shall adopt the notation of ZDN04 and make frequent references to equations and figures therein, generally without explanation or comment.

2 NOTATIONAL REVIEW

For simplicity we consider only single-frequency, fundamental-mode measurements made using untapered seismic recordings in the time domain; the effect of applying either a single or multiple tapers in the measurement process can be easily accounted for using the procedures described in Sections 4 and 9 of ZDN04. The perturbations in the frequency-dependent phase $\delta\phi(\omega)$ and logarithmic amplitude $\delta \ln A(\omega)$ of a surface wave are related to the 3-D velocity and density perturbations $\delta\alpha/\alpha$, $\delta\beta/\beta$ and $\delta\rho/\rho$ by eqs (3.8) of ZDN04:

$$\delta\phi = \iiint_{\oplus} \left[K_\phi^\alpha(\delta\alpha/\alpha) + K_\phi^\beta(\delta\beta/\beta) + K_\phi^\rho(\delta\rho/\rho)\right] d^3\mathbf{x}, \tag{1}$$

$$\delta \ln A = \iiint_{\oplus} \left[K_A^\alpha(\delta\alpha/\alpha) + K_A^\beta(\delta\beta/\beta) + K_A^\rho(\delta\rho/\rho)\right] d^3\mathbf{x}. \tag{2}$$

The 3-D phase and amplitude sensitivity kernels $K_\phi^{\alpha,\,\beta,\,\rho}$ $(\mathbf{x}, \omega)$ and $K_A^{\alpha,\,\beta,\,\rho}$ $(\mathbf{x}, \omega)$ are given by ZDN04 eqs (3.9) and (3.10):

$$K_\phi^{\alpha,\,\beta,\,\rho} = -\mathrm{Im}\left(\frac{\mathcal{S}'\Omega^{\alpha,\,\beta,\,\rho}\mathcal{R}'' e^{-i[k(\Delta'+\Delta''-\Delta)-(n'+n''-n)\pi/2+\pi/4]}}{\mathcal{S}\mathcal{R}\sqrt{8\pi k\,|\sin\Delta'|\,|\sin\Delta''|/|\sin\Delta|}}\right), \tag{3}$$

$$K_A^{\alpha,\,\beta,\,\rho} = \mathrm{Re}\left(\frac{\mathcal{S}'\Omega^{\alpha,\,\beta,\,\rho}\mathcal{R}'' e^{-i[k(\Delta'+\Delta''-\Delta)-(n'+n''-n)\pi/2+\pi/4]}}{\mathcal{S}\mathcal{R}\sqrt{8\pi k\,|\sin\Delta'|\,|\sin\Delta''|/|\sin\Delta|}}\right), \tag{4}$$

where $k(\omega)$ is the wavenumber measured in radians per second on the unit sphere, Δ is the angular epicentral distance, n is the number of polar passages, and we have ignored higher-mode coupling for the reasons articulated by ZDN04 and Zhou *et al.* (2005). The prime and double prime identify quantities associated with the source-to-scatterer and scatterer-to-receiver great-circle paths, of angular arc lengths Δ' and Δ'' and having n' and n'' polar passages, respectively. The quantities $\mathcal{S}$ and $\mathcal{S}'$ account for the surface-wave radiation pattern at the source, whereas $\mathcal{R}$ and $\mathcal{R}''$ account for the polarization of the receiver. The scattering factors $\Omega^{\alpha,\beta,\rho}$ are a measure of the strength of the self-scattering off a 3-D elastic heterogeneity $\delta\alpha/\alpha$, $\delta\beta/\beta$ or $\delta\rho/\rho$ situated at the point $\mathbf{x}$.

By neglecting the angular deflection $\eta = \arccos(\hat{\mathbf{k}}' \cdot \hat{\mathbf{k}}'')$ of a surface wave upon scattering, we can reduce the 3-D dependence of $\delta\phi(\omega)$ and $\delta \ln A(\omega)$ in eqs (1) and (2) to a 2-D dependence upon the local fractional phase-velocity perturbation:

$$\delta\phi = \iint_{\Omega} K_{\phi}^{c}(\delta c/c)\, d\Omega, \qquad \delta \ln A = \iint_{\Omega} K_{A}^{c}(\delta c/c)\, d\Omega, \tag{5}$$

where the integration is over the unit sphere $\Omega = \{\hat{\mathbf{r}} : \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = 1\}$. The 2-D phase-velocity kernels $K_{\phi}^{c}(\hat{\mathbf{r}}, \omega)$ and $K_{A}^{c}(\hat{\mathbf{r}}, \omega)$ are given by eqs (6.3) and (6.4) of ZDN04:

$$K_{\phi}^{c} = -\frac{2k^{3/2} \sin\left[k(\Delta' + \Delta'' - \Delta) - (n' + n'' - n)\pi/2 + \pi/4\right]}{\sqrt{8\pi \,|\sin\Delta'|\,|\sin\Delta''|/|\sin\Delta|}}, \tag{6}$$

$$K_{A}^{c} = -\frac{2k^{3/2} \cos\left[k(\Delta' + \Delta'' - \Delta) - (n' + n'' - n)\pi/2 + \pi/4\right]}{\sqrt{8\pi \,|\sin\Delta'|\,|\sin\Delta''|/|\sin\Delta|}}, \tag{7}$$

where we have made a forward-propagating ($\mathcal{S}' = \mathcal{S}$ and $\mathcal{R}'' = \mathcal{R}$) as well as a forward-scattering ($\eta = 0$) approximation for convenience in what follows. As we shall see, the 3-D and 2-D sensitivities of group-delay and anelastic attenuation measurements can all be expressed in terms of the phase-delay and geometrical attenuation kernels $K_{\phi}^{\alpha,\beta,\rho}(\mathbf{x}, \omega)$, $K_{\phi}^{c}(\hat{\mathbf{r}}, \omega)$ and $K_{A}^{c}(\hat{\mathbf{r}}, \omega)$ given in eqs (3), (6) and (7).

3 3-D GROUP-DELAY SENSITIVITY KERNEL

The perturbation $\delta t(\omega)$ in the group delay of a surface wave is related to the perturbation $\delta\phi(\omega)$ in the phase delay by

$$\delta t = \frac{d(\delta\phi)}{d\omega}. \tag{8}$$

The phase delay $\delta\phi(\omega)$ is measured in radians whereas the group delay $\delta t(\omega)$ is measured in seconds. Since the model perturbations $\delta\alpha/\alpha$, $\delta\beta/\beta$ and $\delta\rho/\rho$ are independent of the angular frequency ω, differentiation of eq. (1) immediately yields the desired 3-D group-delay kernel:

$$\delta t = \iiint_{\oplus} \left[K_{t}^{\alpha}(\delta\alpha/\alpha) + K_{t}^{\beta}(\delta\beta/\beta) + K_{t}^{\rho}(\delta\rho/\rho)\right] d^{3}\mathbf{x}, \tag{9}$$

where

$$K_{t}^{\alpha,\beta,\rho} = \frac{\partial K_{\phi}^{\alpha,\beta,\rho}}{\partial\omega}. \tag{10}$$

We have made no attempt to evaluate the derivative in eq. (10) analytically; see Gilbert (1976) for a related but simpler problem. However, we have found it straightforward to compute $K_{t}^{\alpha,\beta,\rho}(\mathbf{x}, \omega)$ with sufficient accuracy for the purposes of inversion using a simple numerical first-difference method. The lower half of Fig. 1 shows an illustrative example of a group-delay, shear-velocity kernel $K_{t}^{\beta}(\mathbf{x}, \omega)$ for a 10 mHz Love wave; the corresponding phase-delay kernel $K_{\phi}^{\beta}(\mathbf{x}, \omega)$ is plotted above for comparison. The off-great-circle sidebands are more pronounced for $K_{t}^{\beta}(\mathbf{x}, \omega)$ than for $K_{\phi}^{\beta}(\mathbf{x}, \omega)$; this enhanced off-path sensitivity of $\delta t(\omega)$ is consistent with the analytical 2-D Gaussian beam analysis of Nolet & Dahlen (2000). In addition, the group-delay sensitivity is slightly more compressed toward the Earth's surface than the phase-delay sensitivity; however, this is a minor effect compared to the higher sidebands and difficult to discern on the scale of the AB slice views in Fig. 1.

4 2-D GROUP-DELAY KERNELS

We can likewise determine the 2-D sensitivity of a group-delay measurement $\delta t(\omega)$ by differentiation of the first of eqs (5). In this case we must be cognizant of the fact that both the 2-D phase-delay kernel K_{ϕ}^{c} and the perturbation $\delta c/c$ depend upon the angular frequency ω:

$$\delta t = \iint_{\Omega} \left[\left(\partial K_{\phi}^{c}/\partial\omega\right)(\delta c/c) + K_{\phi}^{c}\,\frac{\partial}{\partial\omega}(\delta c/c)\right] d\Omega. \tag{11}$$

The derivative of the fractional phase-velocity perturbation $\delta c/c$ is related to the fractional group-velocity perturbation $\delta c/c$ by eq. (22) of Spetzler *et al.* (2002):

$$\frac{\partial}{\partial\omega}\left(\frac{\delta c}{c}\right) = \frac{1}{kC}\left(\frac{\delta C}{C} - \frac{\delta c}{c}\right). \tag{12}$$

The only frequency dependence of the kernel $K_{\phi}^{c}(\hat{\mathbf{r}}, \omega)$ is via the wavenumber $k(\omega)$, which appears in the phase multiplying the angular detour distance $\Delta' + \Delta'' - \Delta$ and as a $k^{3/2}$ pre-factor in eq. (6). Noting that $dk/d\omega = C^{-1}$ and making use of the relation (12) we find—to our initial

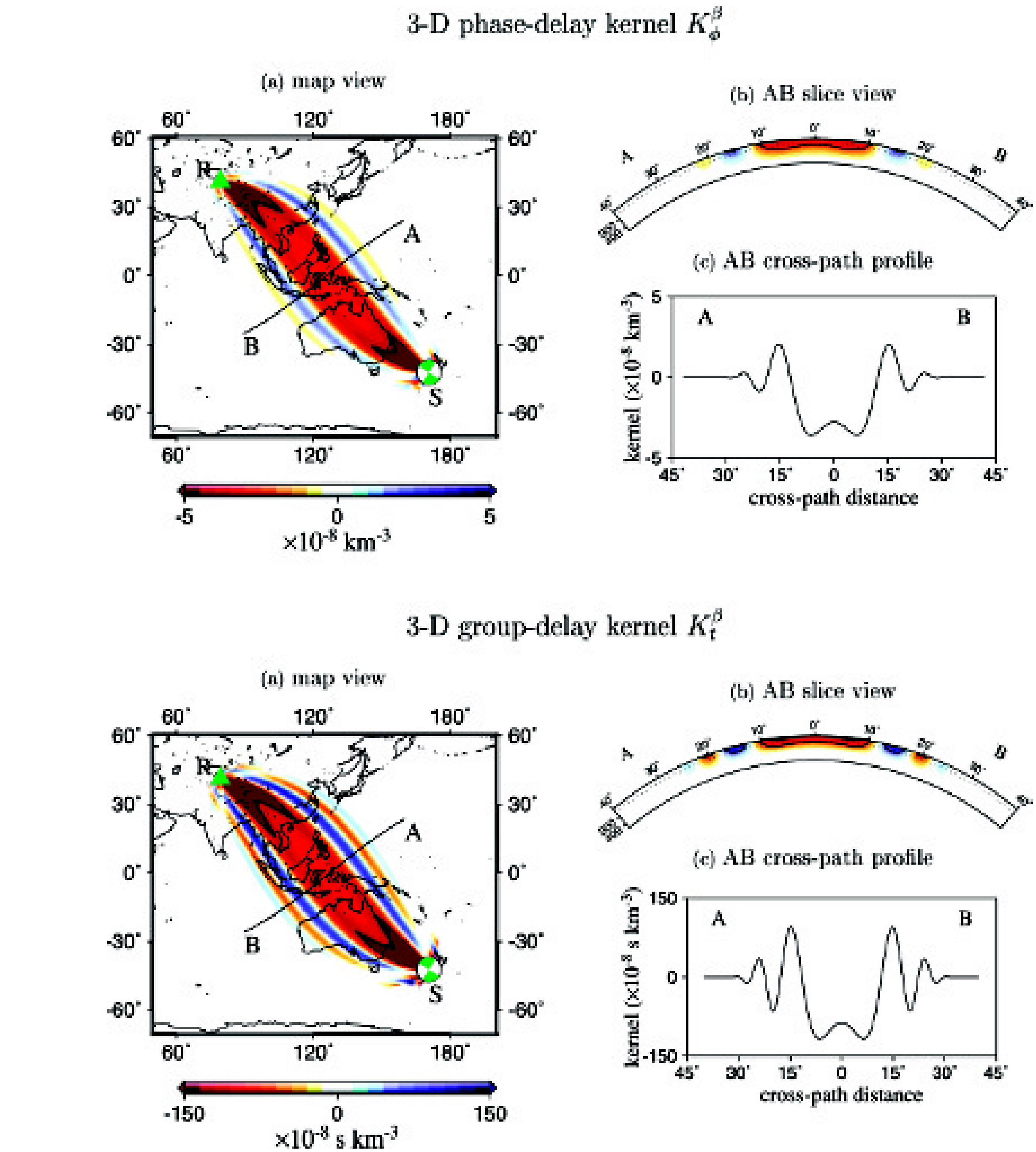

Figure 1. Various views of the 3-D phase-delay kernel $K_\phi^\beta(\mathbf{x}, \omega)$ (top three plots) and the 3-D group-delay kernel $K_t^\beta(\mathbf{x}, \omega)$ (bottom three plots) for a 10 mHz, fundamental-mode Love wave. (a) Map views plotted at 100 km depth. (b) Vertical slice views of cross-section AB, midway between the source and receiver; dotted lines are plotted at 100 km depth. (c) Variation of the shear-velocity sensitivity along profile AB at 100 km depth. The seismic source is a vertical strike-slip fault (green beachball) situated at a depth of 10 km, with maximum Love-wave radiation in the direction of propagation to the receiver (green triangle). The sensitivity kernel is for a transverse-component measurement made on a cosine-tapered seismogram of 520 s duration, centred upon the theoretical group arrival time in the reference spherical earth model 1066A (Gilbert & Dziewonski 1975).

consternation and in disagreement with eqs (23)–(24) of Spetzler *et al.* (2002)—that the group delay $\delta t(\omega)$ depends upon *both* the fractional group- and phase-velocity perturbations:

$$\delta t = \iint_\Omega K_t^C(\delta C/C)\, d\Omega + \iint_\Omega K_t^c(\delta c/c)\, d\Omega, \tag{13}$$

where

$$K_t^C = \left(\frac{1}{kC}\right) K_\phi^c, \qquad K_t^c = \left(\frac{1}{2kC}\right) K_\phi^c + \left(\frac{\Delta' + \Delta'' - \Delta}{C}\right) K_A^c. \tag{14}$$

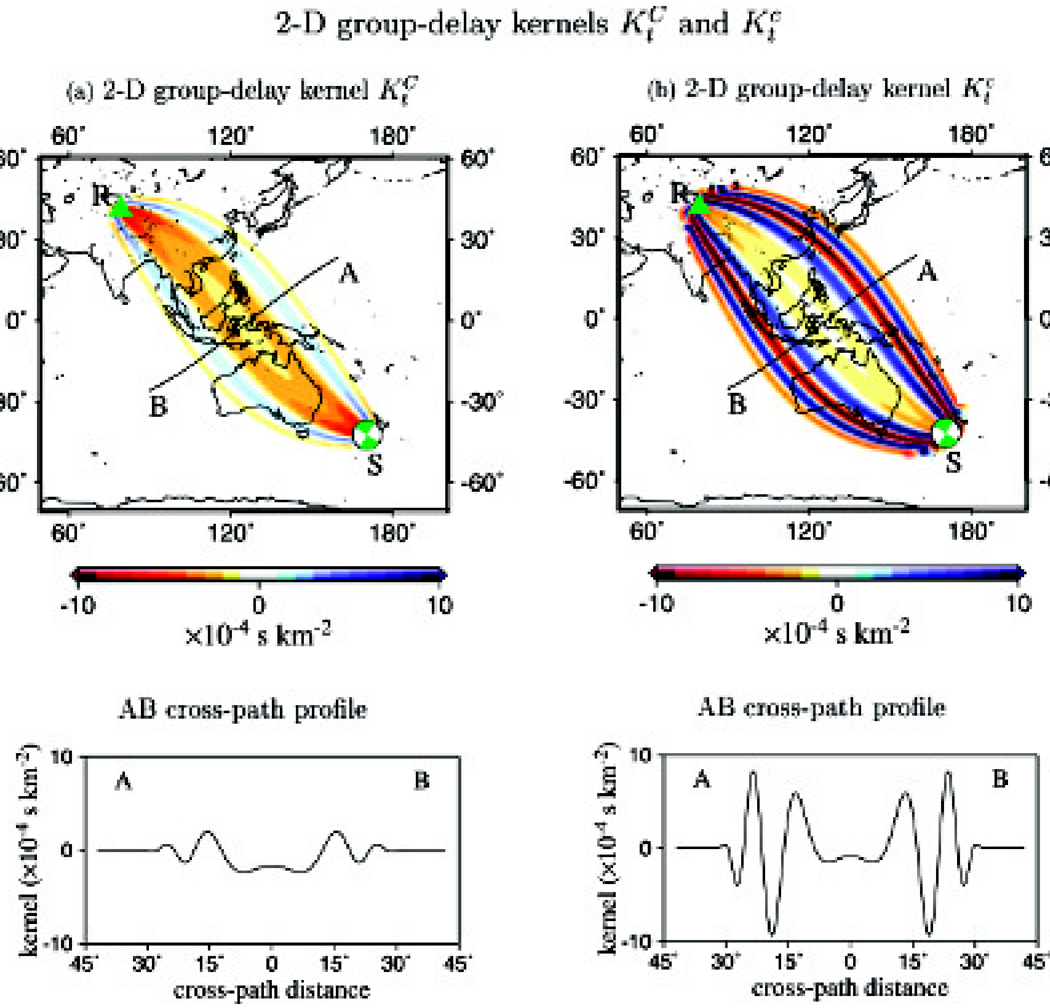

Figure 2. (a) Kernel $K_t^C(\hat{\mathbf{r}}, \omega)$ expressing the sensitivity of a 10 mHz Love-wave group delay $\delta t(\omega)$ to 2-D variations in the local fractional group velocity $\delta C/C$. (b) Corresponding 2-D kernel $K_t^c(\hat{\mathbf{r}}, \omega)$ expressing the simultaneous sensitivity to the local phase velocity $\delta c/c$. Top plots show map views for the same source–receiver path as in Fig. 1; bottom plots show variation along the perpendicular cross section AB, midway between the source and receiver. The measurements are presumed to be made on a cosine-tapered seismogram of 520 s duration, centred upon the group arrival time in model 1066A (Gilbert & Dziewonski 1975). The strong sideband sensitivity to short-scale phase-velocity variations $\delta c/c$ is the most remarkable feature of the plots.

In retrospect, this dual dependence upon both $\delta C/C$ and $\delta c/c$ is not that surprising, inasmuch as $\delta t(\omega)$ is the perturbation in the arrival time of a group of constructively and destructively interfering waves, each of which is propagating at its own local phase velocity $c(\hat{\mathbf{r}}, \omega) + \delta c(\hat{\mathbf{r}}, \omega)$.

Fig. 2 shows an illustrative plot of the two kernels $K_t^C(\hat{\mathbf{r}}, \omega)$ and $K_t^c(\hat{\mathbf{r}}, \omega)$ for a 10 mHz Love wave. It is noteworthy that the sensitivity to the fractional phase velocity $\delta c/c$ actually exceeds the sensitivity to the group velocity $\delta C/C$, by as much as a factor of four in the sidebands beyond the first Fresnel zone. It would obviously be unwise to use eqs (13)–(14) as a basis for finite-frequency inversion of measured group delays $\delta t(\omega)$ to obtain 2-D maps of group velocity $\delta C/C$ without simultaneous consideration of the even stronger dependence upon the phase velocity $\delta c/c$.

5 REFORMULATION OF THE DUAL DEPENDENCE

In an exemplary review of the originally submitted version of this paper, Mike Ritzwoller has pointed out to us that the strong 2-D dependence of $\delta t(\omega)$ upon the phase velocity can be ameliorated by using eq. (12) to rewrite eqs (13)–(14) in the alternative form

$$\delta t = \iint_\Omega \tilde{K}_t^C(\delta C/C)\, d\Omega + \iint_\Omega \tilde{K}_t^c\left[\omega \frac{\partial}{\partial\omega}(\delta c/c)\right] d\Omega, \tag{15}$$

where

$$\tilde{K}_t^C = K_t^C + K_t^c = \left(\frac{3}{2kC}\right) K_\phi^c + \left(\frac{\Delta' + \Delta'' - \Delta}{C}\right) K_A^c, \tag{16}$$

$$\tilde{K}_t^c = -\left(\frac{C}{c}\right) K_t^c = -\left(\frac{1}{2\omega}\right) K_\phi^c - \left(\frac{\Delta' + \Delta'' - \Delta}{c}\right) K_A^c. \tag{17}$$

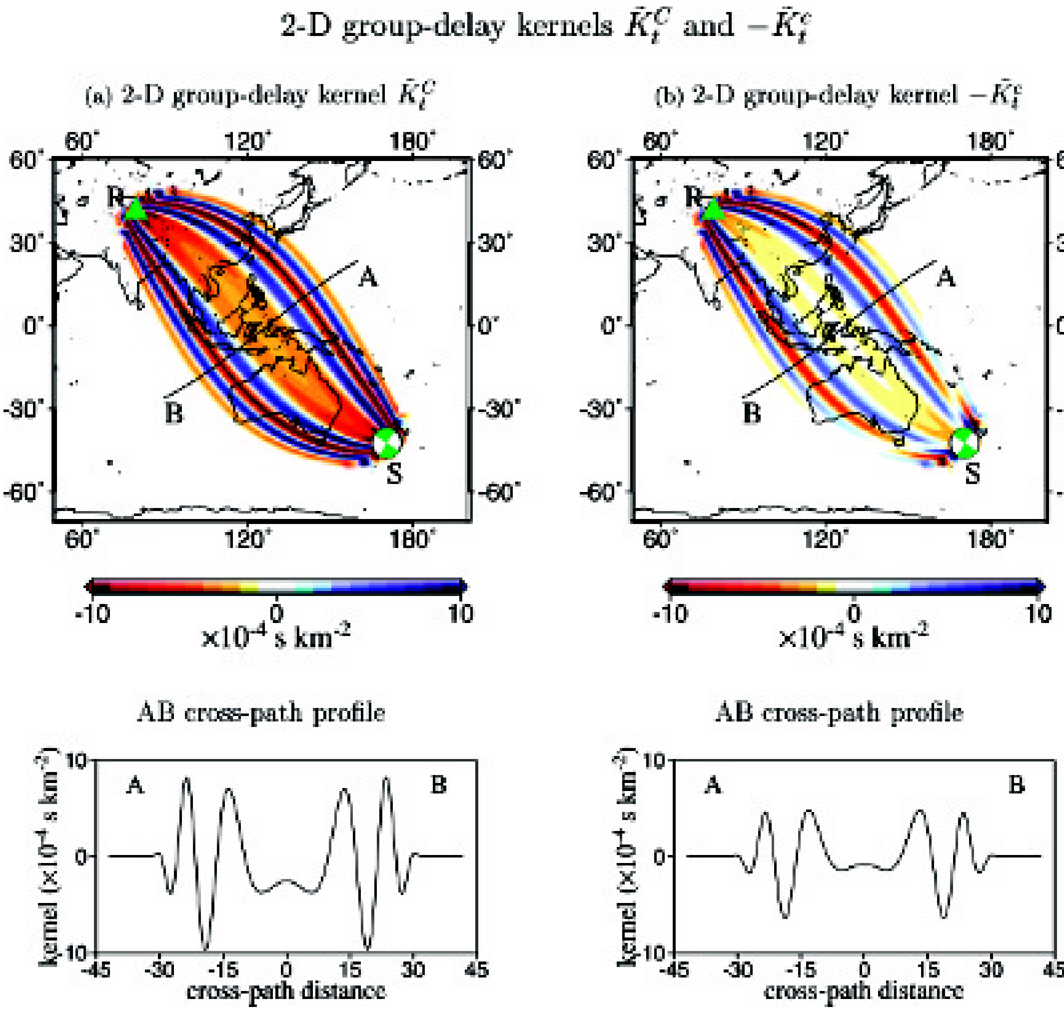

Figure 3. Same as Fig. 2, except for the reformulated 2-D kernels $\tilde{K}_t^C(\hat{\mathbf{r}}, \omega)$ and $\tilde{K}_t^c(\hat{\mathbf{r}}, \omega)$ expressing the sensitivity to $\delta C/C$ and $\omega(\partial/\partial\omega)(\delta c/c)$, respectively. The two reformulated kernels have opposite signs within the first Fresnel zone, and we have plotted $-\tilde{K}_t^c(\hat{\mathbf{r}}, \omega)$ rather than $\tilde{K}_t^c(\hat{\mathbf{r}}, \omega)$ to facilitate comparison. Both kernels are characterized by strong sidebands, arising from the multiplicative factor $\Delta' + \Delta'' - \Delta$ in eqs (16) and (17).

Eqs (15)–(17) express the dual dependence not directly upon the phase velocity $\delta c/c$ but rather upon its dimensionless dispersion $\omega(\partial/\partial\omega)(\delta c/c)$. This has the advantage that the sensitivity to $\delta C/C$ is greater than that to $\omega(\partial/\partial\omega)(\delta c/c)$, particularly in the central Fresenl zone, where the amplitude of the reformulated kernel $\tilde{K}_t^c(\hat{\mathbf{r}}, \omega)$ exceeds that of $-\tilde{K}_t^c(\hat{\mathbf{r}}, \omega)$ by approximately a factor of three, as illustrated in Fig. 3. Because of this, and because the amplitude of the group-velocity perturbation $\delta C/C$ exceeds that of the phase-velocity dispersion $\omega(\partial/\partial\omega)(\delta c/c)$ by roughly the same factor in the current generation of smooth, upper-mantle 3-D models, Barmin *et al.* (2005) have suggested that it is permissible simply to ignore the dependence upon the phase velocity and approximate the sensitivity of a measured group delay by the first term in eq. (15). Such an approximation may not be unreasonable for the largest-scale lateral variations in the group velocity $\delta C/C$; however, any attempt to resolve small-scale structure in $\delta C/C$ is likely to be plagued by the strong sidebands of the reformulated 2-D kernel $\tilde{K}_t^C(\hat{\mathbf{r}}, \omega)$. We believe that it is preferable to eschew an intermediate 2-D inversion for $\delta C/C$, and instead to invert measured group delays $\delta t(\omega)$ directly for 3-D structural variations in the *S*-wave velocity $\delta\beta/\beta$, using eqs (9)–(10).

6 REDUCTION TO RAY THEORY

If the lateral variations in $\delta C/C$, $\delta c/c$ and $\omega(\partial/\partial\omega)(\delta c/c)$ are sufficiently smooth across the great-circle ray path, these factors can be extracted from the cross-path integrals in eqs (13) and (15). The remaining cross-path integrals of the Fréchet kernels $K_t^C(\hat{\mathbf{r}}, \omega)$, $K_t^c(\hat{\mathbf{r}}, \omega)$ and $\tilde{K}_t^C(\hat{\mathbf{r}}, \omega)$, $\tilde{K}_t^c(\hat{\mathbf{r}}, \omega)$ can be performed analytically by making the paraxial approximation

$$\Delta' + \Delta'' - \Delta \approx \frac{1}{2}\left[\frac{\sin\Delta}{\sin x\,\sin(\Delta - x)}\right]y^2, \tag{18}$$

where x and y are the along-path and cross-path angular coordinates, and we have restricted attention to a minor-arc wave, as in Section 7 of ZDN04, for simplicity. Upon making use of the Gaussian integral identities in eq. (7.13) of ZDN04 we find that

$$\int_{-\infty}^{\infty} K_t^C\, dy = \int_{-\infty}^{\infty} \tilde{K}_t^C\, dy = -1/C, \qquad \int_{-\infty}^{\infty} K_t^c\, dy = \int_{-\infty}^{\infty} \tilde{K}_t^c\, dy = 0, \tag{19}$$

so that the dependencies upon the phase velocity $\delta c/c$ or its dispersion $\omega(\partial/\partial\omega)(\delta c/c)$ vanish, and we recover the 1-D, ray-theoretical dependence upon the along-ray group-velocity variations,

$$\delta t \approx -C^{-2} \int_0^{\Delta} \delta C\, dx, \tag{20}$$

in this infinite-frequency limit, as expected. The Fermat path-integral relation (20) has been used as the basis for making 2-D maps of $\delta C/C$ in a number of regional group-velocity investigations. Because of the strong sidebands of the 2-D kernels $K_t^C(\hat{\mathbf{r}}, \omega)$ and, especially, $K_t^c(\hat{\mathbf{r}}, \omega)$, $\tilde{K}_t^C(\hat{\mathbf{r}}, \omega)$ and $\tilde{K}_t^c(\hat{\mathbf{r}}, \omega)$, the cross-path variations of $\delta C/C$, $\delta c/c$ and $\omega(\partial/\partial\omega)(\delta c/c)$ need to be smooth out to a considerable distance off the great-circle ray path for the ray-theoretical approximation (20) to be valid.

7 3-D ANELASTIC ATTENUATION KERNELS

Starting with an unperturbed, perfectly elastic, spherical earth model with bulk and shear moduli κ and μ, we next consider a purely imaginary 3-D perturbation of the form

$$\frac{\delta\kappa}{\kappa} = iQ_\kappa^{-1}, \qquad \frac{\delta\mu}{\mu} = iQ_\mu^{-1}, \qquad \frac{\delta\rho}{\rho} = 0, \tag{21}$$

or, equivalently,

$$\frac{\delta\alpha}{\alpha} = \frac{i}{2}\left(1 - \frac{4}{3}\frac{\beta^2}{\alpha^2}\right)Q_\kappa^{-1} + \frac{i}{2}\left(\frac{4}{3}\frac{\beta^2}{\alpha^2}\right)Q_\mu^{-1}, \qquad \frac{\delta\beta}{\beta} = \frac{i}{2}Q_\mu^{-1}, \qquad \frac{\delta\rho}{\rho} = 0, \tag{22}$$

where Q_κ^{-1} and Q_μ^{-1} are the spatially variable, inverse bulk and shear quality factors. Upon inserting eqs (22) into eq. (2) and rearranging terms, we obtain the Fréchet derivative relationship expressing the 3-D sensitivity of a measured amplitude perturbation $\delta \ln A(\omega)$ to the inverse quality factors:

$$\delta \ln A = \iiint_{\oplus} \left[K_A^{Q_\kappa}\, Q_\kappa^{-1} + K_A^{Q_\mu}\, Q_\mu^{-1}\right] d^3\mathbf{x}, \tag{23}$$

where

$$K_A^{Q_\kappa} = \frac{1}{2}\left(1 - \frac{4}{3}\frac{\beta^2}{\alpha^2}\right)K_\phi^\alpha, \qquad K_A^{Q_\mu} = \frac{1}{2}\left(K_\phi^\beta + \frac{4}{3}\frac{\beta^2}{\alpha^2}K_\phi^\alpha\right). \tag{24}$$

It is noteworthy that the anelastic attenuation kernels $K_A^{Q_\kappa}(\mathbf{x},\omega)$ and $K_A^{Q_\mu}(\mathbf{x},\omega)$ are linear combinations of the elastic phase-delay kernels $K_\phi^\alpha(\mathbf{x}, \omega)$ and $K_\phi^\beta(\mathbf{x}, \omega)$ rather than the geometrical attenuation kernels $K_A^\alpha(\mathbf{x}, \omega)$ and $K_A^\beta(\mathbf{x}, \omega)$.

The results in eq. (24) can be rewritten in a form analogous to eq. (3), namely

$$K_A^{Q_\kappa,\, Q_\mu} = -\mathrm{Im}\left(\frac{\mathcal{S}'\Omega^{Q_\kappa,\, Q_\mu}\mathcal{R}'' e^{-i[k(\Delta'+\Delta''-\Delta)-(n'+n''-n)\pi/2+\pi/4]}}{\mathcal{S}\mathcal{R}\sqrt{8\pi k\,|\sin\Delta'|\,|\sin\Delta''|/|\sin\Delta|}}\right), \tag{25}$$

where

$$\Omega^{Q_\kappa} = \begin{cases} 0 & \text{Love waves} \\ -\kappa(\dot{U} + 2r^{-1}U - kr^{-1}V)^2 & \text{Rayleigh waves,} \end{cases} \tag{26}$$

$$\Omega^{Q_\mu} = \begin{cases} -\mu\left[(\dot{W} - r^{-1}W)^2\cos\eta + k^2r^{-2}W^2\cos 2\eta\right] & \text{Love waves} \\ -\mu\left[\tfrac{1}{3}(2\dot{U} - 2r^{-1}U + kr^{-1}V)^2 \right. & \\ \qquad \left. + (\dot{V} - r^{-1}V + kr^{-1}U)^2\cos\eta + k^2r^{-2}V^2\cos 2\eta\right] & \text{Rayleigh waves.} \end{cases} \tag{27}$$

The anelastic self-scattering factors (26)–(27) are analogous to the corresponding elastic scattering factors $\Omega^{\alpha,\beta,\rho}$ tabulated in Appendix A of ZDN04; the quantities U, V and W are the Rayleigh and Love eigenfunctions, normalized in accordance with eqs (2.11)–(2.12) of ZDN04, and a dot denotes differentiation with respect to radius r.

Love-wave attenuation is independent of the bulk anelasticity Q_κ^{-1} whereas Rayleigh waves are attenuated by both Q_κ^{-1} and Q_μ^{-1}, but much more strongly by the latter. To illustrate this we show examples of $K_A^{Q_\mu}(\mathbf{x},\omega)$ for a 10 mHz Love wave and of both $K_A^{Q_\mu}(\mathbf{x},\omega)$ and $K_A^{Q_\kappa}(\mathbf{x},\omega)$ for a 10 mHz Rayleigh wave in Fig. 4. All three sensitivity kernels are negative within the first Fresnel zone, as expected if a physically permissible inverse quality factor, $Q_\kappa^{-1}, Q_\mu^{-1} > 0$, is to lead to a reduction in the wave amplitude. Roughly speaking, a 10 mHz Rayleigh wave is an order of magnitude less sensitive to bulk anelasticity variations Q_κ^{-1} than to shear anelasticity variations Q_μ^{-1}.

Finally we note that the above results can be written more succinctly in terms of the inverse P-wave and S-wave quality factors, defined by eqs (9.59)–(9.60) of Dahlen & Tromp (1998) and commonly used in body-wave seismology:

$$Q_\alpha^{-1} = \left(1 - \frac{4}{3}\frac{\beta^2}{\alpha^2}\right)Q_\kappa^{-1} + \left(\frac{4}{3}\frac{\beta^2}{\alpha^2}\right)Q_\mu^{-1}, \qquad Q_\beta^{-1} = Q_\mu^{-1}. \tag{28}$$

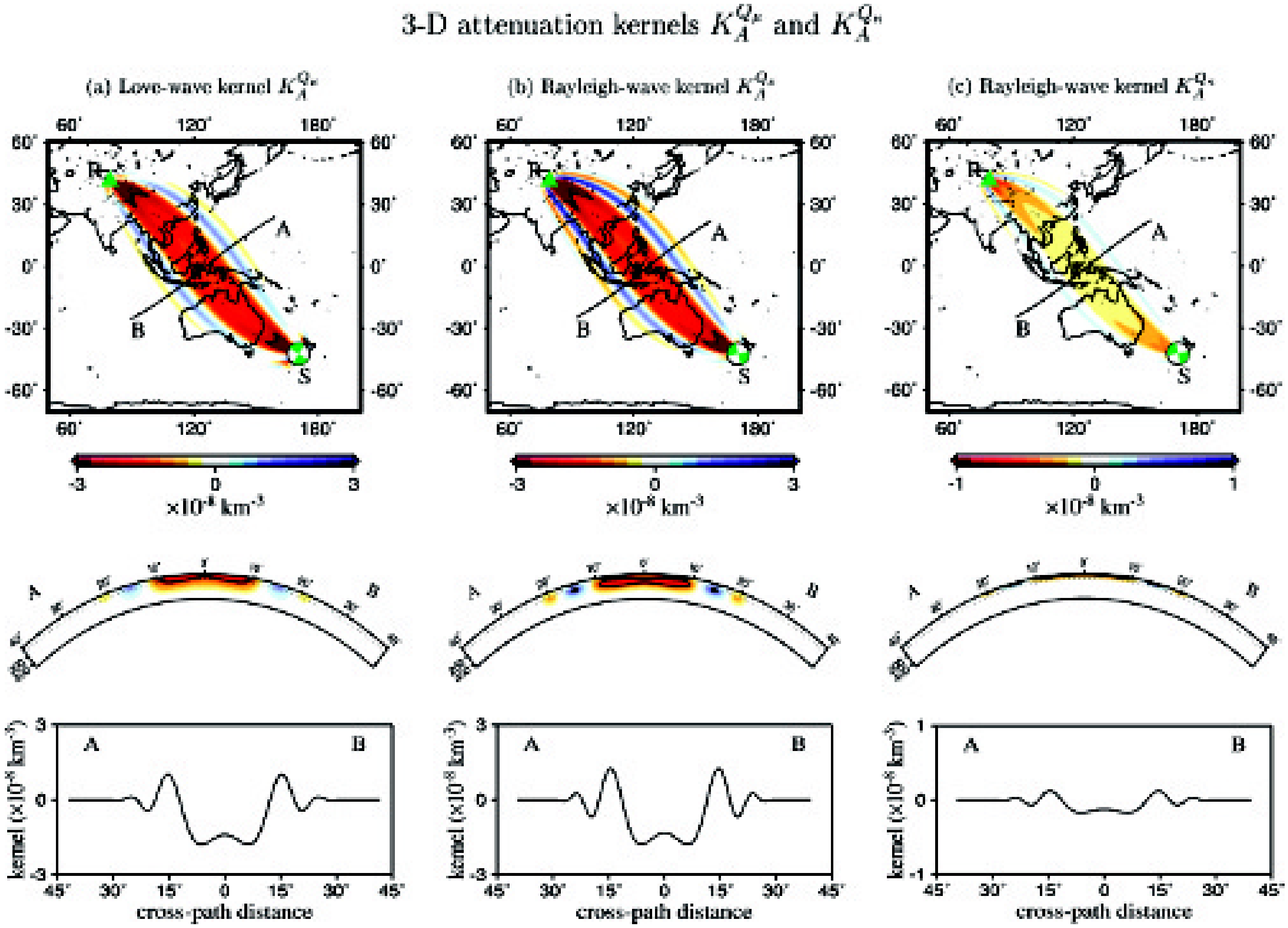

Figure 4. Various views of the 3-D anelastic attenuation kernels for 10 mHz Love and Rayleigh waves. (a) Love-wave kernel $K_A^{Q_\mu}(\mathbf{x},\omega)$, expressing the sensitivity of a measured amplitude perturbation $\delta \ln A(\omega)$ to 3-D variations in the inverse shear quality factor Q_μ^{-1}. (b) Shear attenuation kernel $K_A^{Q_\mu}(\mathbf{x},\omega)$ for a 10 mHz Rayleigh wave. (c) Bulk attenuation kernel $K_A^{Q_\kappa}(\mathbf{x},\omega)$ for the same Rayleigh wave. Top three plots are map views plotted at 100 km depth; middle three plots are vertical slices along cross-section AB with 100 km depth indicated by the dotted line; bottom three plots show variation of $K_A^{Q_\kappa\,\mu}(\mathbf{x},\omega)$ along profile AB at 100 km depth. Both the Love-wave and Rayleigh-wave sources are 10-km-deep, vertical strike slip-faults (green beachballs), with the latter rotated in strike by $\pi/4$ with respect to the former, so that the maximum radiation is in both instances in the direction of propagation to the receiver (green triangle). The Love-wave sensitivity kernel is for a transverse-component measurement and the Rayleigh-wave kernel is for a vertical-component measurement, both made on cosine-tapered seismograms of 520 s duration, centred upon the group arrival time in the reference earth model 1066A (Gilbert & Dziewonski 1975).

In this notation the imaginary velocity perturbations in eq. (22) are simply $\delta\alpha/\alpha = (i/2)Q_\alpha^{-1}$, $\delta\beta/\beta = (i/2)Q_\beta^{-1}$ so that the 3-D Fréchet kernel relationship (23)–(24) reduces to

$$\delta \ln A = \iiint_\oplus \left[K_A^{Q_\alpha}\, Q_\alpha^{-1} + K_A^{Q_\beta}\, Q_\beta^{-1}\right] d^3\mathbf{x} \quad \text{where} \quad K_A^{Q_\alpha,\, Q_\beta} = \frac{1}{2} K_\phi^{\alpha,\,\beta}. \tag{29}$$

8 2-D ATTENUATION KERNEL

A 2-D anelastic sensitivity kernel can be derived by making a forward-scattering ($\eta = 0$) approximation in eqs (26)–(27) and evaluating the resulting integral over depth; the anelastic analogue of eq. (6.2) of ZDN04 is

$$\int_0^a \left[\Omega_{\eta=0}^{Q_\kappa}\, Q_\kappa^{-1} + \Omega_{\eta=0}^{Q_\mu}\, Q_\mu^{-1}\right] r^2\, dr = -k^2\left(\frac{c}{CQ}\right), \tag{30}$$

where Q is the local quality factor of a Love or Rayleigh wave, given by eqs (16.148)–(16.153) of Dahlen & Tromp (1998). Alternatively, it is possible to start with the 2-D amplitude kernel $K_A^c(\hat{\mathbf{r}}, \omega)$ in the second of eqs (5) and note that anelasticity corresponds to an imaginary perturbation in the local fractional phase velocity of the form

$$\frac{\delta c}{c} = \frac{i}{2}\left(\frac{c}{CQ}\right). \tag{31}$$

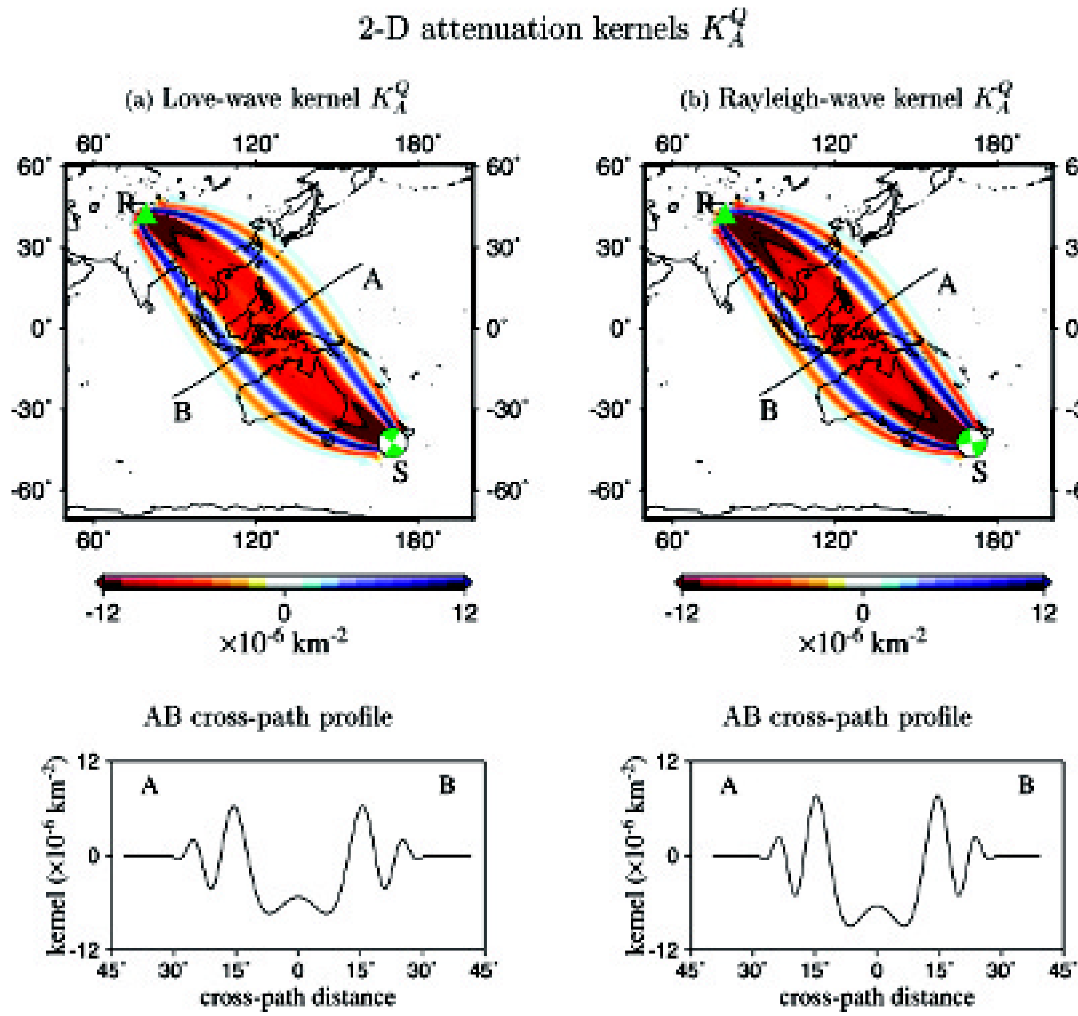

Figure 5. (a) Sensitivity kernel $K_A^Q(\hat{\mathbf{r}}, \omega)$ expressing the sensitivity of $\delta \ln A(\omega)$ to 2-D variations in the inverse quality factor Q^{-1} of a 10 mHz Love wave. (b) Same but for a 10 mHz Rayleigh wave. Top plots show map views and bottom plots show variation along the perpendicular cross section AB, midway between the source and receiver; the amplitude measurements are presumed to be made on cosine-tapered seismograms of 520 s duration, as in Figs 1–4.

Either method yields the same 2-D sensitivity to the inverse surface-wave quality factor Q^{-1}, namely

$$\delta \ln A = \iint_{\Omega} K_A^Q \, Q^{-1} \, d\Omega \quad \text{where} \quad K_A^Q = \left(\frac{c}{2C}\right) K_\phi^c. \tag{32}$$

Just as the 3-D kernels $K_A^{Q_\kappa,\,Q_\mu}(\mathbf{x},\omega)$ and $K_A^{Q_\alpha,\,Q_\beta}(\mathbf{x},\omega)$ are linear combinations of $K_\phi^{\alpha,\,\beta}(\mathbf{x}, \omega)$, the 2-D kernel $K_A^Q(\hat{\mathbf{r}}, \omega)$ is simply a frequency-dependent constant $c/(2C)$ times the 2-D phase-delay kernel $K_\phi^c(\hat{\mathbf{r}}, \omega)$. In Fig. 5 we show illustrative examples of the 2-D anelastic attenuation kernels $K_A^Q(\hat{\mathbf{r}}, \omega)$ along the same source–receiver path as in Figs 1–4. The 2-D sensitivity of a 10 mHz Rayleigh wave is slightly higher than that of a 10 mHz Love wave because it has a slightly larger wavenumber: $(k_{\mathrm{R}}/k_{\mathrm{L}})^{3/2} \approx 1.3$.

9 REDUCTION TO RAY THEORY REDUX

In the limit of infinite frequency, $\omega \to \infty$, the local inverse quality factor Q^{-1} can be extracted from the cross-path integral in eq. (32), and the remaining cross-path integral over the 2-D kernel $K_A^Q(\hat{\mathbf{r}}, \omega)$ can be evaluated by making the paraxial approximation (18), with the result

$$\int_{-\infty}^{\infty} K_A^Q \, dy = -\frac{\omega}{2C}. \tag{33}$$

In this limit we recover the anelastic ray-theoretical result,

$$\delta \ln A \approx -\frac{\omega}{2C} \int_0^{\Delta} \frac{dx}{Q}, \tag{34}$$

as required for the finite-frequency theory to be consistent. A factor of C rather than c appears in the denominators of eqs (31) and (34) because the energy of a dispersive wave propagates with the group velocity.

10 GEOMETRICAL PLUS ANELASTIC ATTENUATION

On a realistic earth model with 3-D variations in $\delta\alpha/\alpha$, $\delta\beta/\beta$ and $\delta\rho/\rho$ as well as Q_κ^{-1} and Q_μ^{-1}, the amplitude of a surface wave will be perturbed by elastic focusing and defocusing effects as well as by anelastic attenuation. The total first-order amplitude perturbation is the sum of both effects:

$$\delta \ln A = \delta \ln A_{\mathrm{el}} + \delta \ln A_{\mathrm{an}}, \tag{35}$$

where $\delta \ln A_{\mathrm{el}}(\omega)$ is given by eqs (2) and (4), whereas $\delta \ln A_{\mathrm{an}}(\omega)$ is given by eqs (23)–(27). In the 2-D, forward-scattering, forward-propagating approximation, the amplitude depends only upon the local surface-wave phase-velocity and inverse quality factor:

$$\delta \ln A = \iint_\Omega K_A^c(\delta c/c)\, d\Omega + \iint_\Omega K_A^Q\, Q^{-1}\, d\Omega. \tag{36}$$

Finally, in the ray-theoretical limit, $\omega \to \infty$, the amplitude is the sum of two 1-D integrals along the unperturbed great-circle ray path:

$$\delta \ln A = -\frac{1}{2c\sin\Delta}\int_0^\Delta \sin x\, \sin(\Delta - x)\, \partial_y^2\, \delta c\, dx - \frac{\omega}{2C}\int_0^\Delta \frac{dx}{Q}. \tag{37}$$

The spherical-earth, elastic focusing-defocusing term, eq. (7.12) of ZDN04, was first derived using a strictly ray-theoretical argument by Woodhouse & Wong (1986).

11 CONCLUSION

In this paper we have derived Fréchet kernels expressing the linearized sensitivity of a surface-wave group-delay measurement $\delta t(\omega)$ to 3-D elastic velocity and density variations $\delta\alpha/\alpha$, $\delta\beta/\beta$ and $\delta\rho/\rho$ and the sensitivity of an amplitude measurement $\delta \ln A(\omega)$ to 3-D bulk and shear anelasticity variations Q_κ^{-1} and Q_μ^{-1}. By making a forward-scattering ($\eta = 0$) and a forward-propagating ($\mathcal{S}' = \mathcal{S}$ and $\mathcal{R}'' = \mathcal{R}$) approximation and evaluating the resulting integrals over depth, we have implemented a reduction from 3-D to 2-D, obtaining kernels that express the sensitivity of $\delta t(\omega)$ to the local fractional group-velocity and phase-velocity perturbations $\delta C/C$, $\delta c/c$ and $\omega(\partial/\partial\omega)(\delta c/c)$, and the sensitivity of $\delta \ln A(\omega)$ to the local surface-wave inverse quality factor Q^{-1}. In the ray-theoretical limit, $\omega \to \infty$, these 2-D relationships reduce in turn to the expected 1-D, along-ray integrations. The strong sensitivity of a measured group delay $\delta t(\omega)$ to short-scale, off-path variations in the phase velocity $\delta c/c$ could potentially give rise to serious artefacts in 2-D maps of $\delta C/C$ obtained by ray-theoretical, group-delay inversion. In our opinion it is strongly preferable to invert both group-delay measurements $\delta t(\omega)$ and anelastic attenuation measurements $\delta \ln A(\omega)$ using the full 3-D, finite-frequency sensitivity kernels $K_t^{\alpha,\beta,\rho}(\mathbf{x}, \omega)$ and $K_A{}^{Q_\kappa, Q_\mu}(\mathbf{x},\omega)$.

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